Hyper-complex four-manifolds from the Tzitzéica equation

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Abstract

It is shown how solutions to the Tzitzéica equation can be used to construct a family of (pseudo) hyper-complex metrics in four dimensions.

1 Introduction

A striking *universal* feature of integrable systems is that the same integrable equations often arise from many unrelated sources. The Tzitzéica equation [11]

$$\omega_{xy} = e^{\omega} - e^{-2\omega} \tag{1}$$

is a good example. It first arose in a study of surfaces in $\mathbb{R}^3$ for which the ratio of the negative Gaussian curvature to the fourth power of a distance from a tangent plane to some fixed point is a constant. Tzitzéica has shown that if x and y are coordinates on such a surface in which the second fundamental form is off-diagonal, then there exists a real function $\omega(x, y)$ such that the Peterson-Codazzi equations reduce to (1). Moreover, he has demonstrated [12] that (1) is a consistency condition for an otherwise overdetermined system of PDEs[1] for $\psi_i(x, y),\ i = 1, 2, 3.$

$$\begin{aligned}
\partial_x \begin{pmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\psi}_3 \end{pmatrix} &= \begin{pmatrix} -\omega_x & 0 & \lambda \\ 1 & \omega_x & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\psi}_3 \end{pmatrix}, \\
\partial_y \begin{pmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\psi}_3 \end{pmatrix} &= \begin{pmatrix} 0 & e^{-2\omega} & 0 \\ 0 & 0 & e^{\omega} \\ \lambda^{-1}e^{\omega} & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\psi}_3 \end{pmatrix}.
\end{aligned} \tag{2}$$

The above linear system is in modern terminology known as a 'Lax pair with a spectral parameter'. It underlines the complete integrability of the Tzitzéica equation [10].

Equation (1) reappeared in the context of soliton solutions [4], gas dynamics [7] as well as geometry of affine spheres [9]. In this paper I shall reveal yet another occurrence of (1), and show how its solutions can be used to generate explicit pseudo-hyper-complex structures in four dimensions. This will be done by regarding (2) as a reduced Lax pair for $SL(3, \mathbb{R})$ anti-self-dual Yang-Mills (ASDYM) equations, embedding $SL(3, \mathbb{R})$ in Diff($\mathbb{RP}^2$), and reinterpreting the Lax pair in terms of vector fields on $\mathcal{M} = \mathbb{R}^2 \times \mathbb{RP}^2$. Four independent vector fields in this Lax pair will provide a null frame for a pseudo-hyper-complex conformal structure on $\mathcal{M}$.

In the next section the Lax formulation of the pseudo-hyper-complex condition in four dimensions will be given following [5, 8]. In §§3 the connection with the ASDYM will be established, and the explicit embedding of $\mathbf{sl}(3, \mathbb{R})$ in $\mathbf{diff}(\mathbb{RP}^2)$ will be given. The resulting pseudo-hyper-complex structure will be constructed in §§4. All considerations in this section will be local. Finally §§5 contains the twistor interpretation of the construction.

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[1]Strictly speaking the linear system given by Tzitzéica consisted of three second order PDEs for one function. These three equations can be recovered from (2) if one eliminates ψ_1 and ψ_2 by cross-differentiating.

2 Pseudo hyper-complex structures

A smooth real $4n$-dimensional manifold $\mathcal{M}$ equipped with three real endomorphisms $I, S, T : T\mathcal{M} \to T\mathcal{M}$ of the tangent bundle satisfying the algebra of pseudo-quaternions

$$-I^2 = S^2 = T^2 = 1, \qquad IST = 1,$$

is called pseudo-hyper-complex iff the almost complex structure

$$\mathcal{J}_\lambda = aI + bS + cT \tag{3}$$

is integrable for any point of the hyperboloid[2] $a^2 - b^2 - c^2 = 1$. This integrability is equivalent to a vanishing of its Nijenhuis tensor

$$N(X_1, X_2) := (\mathcal{J}_\lambda)^2[X_1, X_2] - \mathcal{J}_\lambda[\mathcal{J}_\lambda X_1, X_2] - \mathcal{J}_\lambda[X_1, \mathcal{J}_\lambda X_2] + [\mathcal{J}_\lambda X_1, \mathcal{J}_\lambda X_2]$$

for arbitrary vectors X_1 and X_2. A convenient matrix representation of the canonical pseudo-hyper-complex structure on $\mathbb{R}^4$ is given by

$$I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In the general case the components of I, S, T depend smoothly on coordinates on $\mathcal{M}$. The endomorphism I endows $\mathcal{M}$ with the structure of a two-dimensional complex manifold, and S and T determine a pair of transverse null foliations. Let g be a metric of signature $(2n, 2n)$ on $\mathcal{M}$. If $(\mathcal{M}, \mathcal{J}_\lambda)$ is pseudo-hyper-complex and

$$g(TX_1, TX_2) = g(SX_1, SX_2) = -g(IX_1, IX_2) = -g(X_1, X_2)$$

for all vectors X_1, X_2 then the triple $(\mathcal{M},\ \mathcal{J}_\lambda,\ g)$ is called a pseudo-hyper-Hermitian structure.

From now on we shall restrict ourselves to oriented four manifolds, where the notions of pseudo-hyper-complex and pseudo-hyper-Hermitian structures coincide. To see it choose any vector $X \in T\mathcal{M}$, and define a conformal structure $[g]$ of signature $(++--)$, by choosing a conformal frame of vector fields (X, IX, SX, TX). Any $g \in [g]$ is then pseudo-hyper-Hermitian. We shall use the following characterisation of the pseudo-hyper-Hermiticity condition:

Proposition 1 ([5]) *Let (X, Y, U, V) be four independent real vector fields on a four-dimensional real manifold $\mathcal{M}$, and let*

$$L_0 = X - \lambda V, \qquad L_1 = U - \lambda Y, \quad \text{where } \lambda \in \mathbb{CP}^1. \tag{4}$$

If

$$[L_0, L_1] = 0 \tag{5}$$

for every λ, then (X, Y, U, V) is a null tetrad for a pseudo-hyper-Hermitian contravariant metric

$$g = X \otimes Y + Y \otimes X - U \otimes V - V \otimes U$$

on $\mathcal{M}$. Every pseudo-hyper-Hermitian metric arises in this way.

For a future reference we write equations (4) in full:

$$[X, U] = 0, \qquad [Y, V] = 0, \qquad [X, Y] - [U, V] = 0. \tag{6}$$

[2] We identify two sheets of this hyperboloid with two unit discs D_- and D_+, and use λ as a projective coordinate on a Riemann sphere $\mathbb{CP}^1 = D_- + D_+ + S^1$. The coordinate λ plays a role of a complex spectral parameter in the Lax pair (4).

Given the null tetrad (X, Y, U, V) we define the pseudo hyper-complex structure by

$$\begin{array}{rclrclrclrcl} I(X) &=& -V, & I(U) &=& -Y, & I(Y) &=& U, & I(V) &=& X, \\ S(X) &=& V, & S(U) &=& Y, & S(Y) &=& U, & S(V) &=& X, \\ T(X) &=& X, & T(U) &=& U, & T(Y) &=& -Y, & T(V) &=& -V. \end{array} \tag{7}$$

Proposition 1 asserts that integrability of I, S, T is guaranteed by (6). Let $\nu \in \Lambda^4(T^*\mathcal{M})$ be the volume form on $\mathcal{M}$. The covariant metric is conveniently expressed in a dual frame

$$\begin{array}{rcll} e_X &=& \nu(..., Y, U, V), & e_Y = \nu(X, ..., U, V), \\ e_U &=& \nu(X, Y, ..., V), & e_V = \nu(X, Y, U, ...), \end{array}$$

and is given by

$$g = e_X \otimes e_Y + e_Y \otimes e_X - e_U \otimes e_V - e_V \otimes e_U.$$

The result of Boyer [1] originally formulated for hyper-complex manifolds still applies (with some sign alterations) in the $(++--)$ signature: a four-manifold is pseudo-hyper-complex iff there exists a basis $(\Omega_1, \Omega_2, \Omega_3)$ of the space of self-dual two forms Λ^2_+, and a one-form $\mathcal{A}$ (called a Lee form) such that

$$\mathrm{d}\Omega_i = \mathcal{A} \wedge \Omega_i. \tag{8}$$

If we change a representative of a pseudo-conformal structure according to $g \to e^f g$, then

$$\Omega_i \longrightarrow e^f \Omega_i, \qquad \mathcal{A} \to \mathcal{A} + \mathrm{d}f.$$

Therefore if $\mathcal{A}$ is exact, then g is conformally pseudo-hyper-Kähler (Ricci-flat).

3 From the Tzitzéica equation to ASDYM

The idea of looking at integrable systems as reductions of the anti-self-dual Yang-Mills (ASDYM) equations goes back to Ward [14]. In this section the list of possible reductions will be enlarged by showing that (1) arises from the $SL(3, \mathbb{R})$ ASDYM with two commuting translational symmetries. In Subsection §§3.1 the connection matrices will be reinterpreted as vector fields on the projective plane.

Consider the flat metric of signature $(2, 2)$ on $\mathbb{R}^4$, which in double null coordinates $x^a = (x, y, u, v)$ takes a form

$$\mathrm{d}x\mathrm{d}y - \mathrm{d}u\mathrm{d}v,$$

and choose the volume element $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}v$. Let $A \in T^*\mathbb{R}^4 \otimes \mathbf{sl}(3, \mathbb{R})$ be a connection one-form on a real rank-three vector bundle, and let F be its curvature two form. In a local trivialisation $A = A_a \mathrm{d}x^a$ and $F = F_{ab} \mathrm{d}x^a \wedge \mathrm{d}x^b$, where $F_{ab} = [D_a, D_b]$ takes its values in $\mathbf{sl}(3, \mathbb{R})$. Here $D_a = \partial_a - A_a$ is the covariant derivative. The connection is defined up to gauge transformations $A \to h^{-1}Ah - h^{-1}\mathrm{d}h$, where $h \in \mathrm{Map}(\mathbb{R}^4, SL(3, \mathbb{R}))$. The ASDYM equations on A_a are $F = -*F$, or

$$F_{xu} = 0, \qquad F_{xy} - F_{uv} = 0, \qquad F_{yv} = 0.$$

These equations are equivalent to the commutativity of the Lax pair

$$L_0 = D_x - \lambda D_v, \qquad L_1 = D_u - \lambda D_y \tag{9}$$

for every value of the parameter λ.

We shall require that the connection possess two commuting translational symmetries X_1, X_2, which in our coordinates are in $X_1 = \partial_u$ and $X_2 = \partial_v$ directions. The direct calculation shows that the ASDYM equations are solved by the following ansatze for Higgs fields A_u and A_v, and gauge

fields A_x and A_y

$$
\begin{aligned}
A_u &= -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{\omega} & 0 & 0 \end{pmatrix}, \qquad A_v = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
A_x &= -\begin{pmatrix} -\omega_x & 0 & 0 \\ 1 & \omega_x & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_y = -\begin{pmatrix} 0 & e^{-2\omega} & 0 \\ 0 & 0 & e^{\omega} \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{10}
$$

iff $\omega(x, y)$ satisfies the Tzitzéica equation (1). We note that the reduced Lax pair (9) could be obtained directly form (2) multiplying the second equation by λ.

We connect the ASDYM equations and those on a pseudo-hyper-complex four-dimensional metric (5) by considering gauge potentials that take values in a Lie algebra of vector fields on some manifold. Proposition 1 reveals one such connection: We make the identification: $X = D_x,\ Y = D_y,\ U = D_u,\ V = D_v$. By comparing (9) with (4), we see that the pseudo-hyper-complex equation is a reduction of the ASDYM with the infinite-dimensional gauge group $\mathrm{Diff}(\mathcal{M})$ by translations along the four coordinate vectors $\partial_x,\ \partial_y,\ \partial_u,\ \partial_v$.

To reveal the connection with the Tzitzéica equation we shall proceed in a slightly different way: Consider the ASDYM equations with the gauge group G, being a sup-group of $\mathrm{Diff}(\Sigma)$, where Σ is some two-dimensional real manifold. We can represent the components of the connection form of A by vector fields on Σ depending also on the coordinates on $\mathbb{R}^4$. Now we suppose that A is invariant under two translations. The reduced Lax pair will then descend to $\mathcal{M} = \mathbb{R}^2 \times \Sigma$ and give rise to a pseudo-hyper-complex metric. A similar idea have been used in [15, 6] to construct new classes of hyper-Kähler four-manifolds out of solutions to some integrable ODEs and PDEs.

Because we are interested in the case $G = SL(3, \mathbb{R})$, we take Σ to be a real projective plane $\mathbb{RP}^2$ with a natural $PSL(3, \mathbb{R})$ group action. The relevant vector fields will be constructed in the next subsection.

3.1 $\mathbf{sl}(3, \mathbb{R})$ as a sub-algebra of $\mathbf{diff}(\mathbb{RP}^2)$

To construct a null tetrad for a pseudo-hyper-complex metric we will need an explicit embedding $\mathbf{sl}(3, \mathbb{R}) \to \mathbf{diff}(\mathbb{RP}^2)$. Let

$$
\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \in SL(3, \mathbb{R}).
$$

Consider the projective transformations of a plane with local coordinates (p, q):

$$
p \longrightarrow \frac{A_{11}p + A_{12}q + A_{13}}{A_{31}p + A_{32}q + A_{33}}, \qquad q \longrightarrow \frac{A_{21}p + A_{22}q + A_{23}}{A_{31}p + A_{32}q + A_{33}}.
$$

This gives rise to a representation of the Lie algebra $\mathbf{sl}(3, \mathbb{R})$ of $SL(3, \mathbb{R})$ by vector fields on $\mathbb{RP}^2$. The easiest way to obtain this representation is to consider the infinitesimal linear left action of $SL(3, \mathbb{R})$ on $\mathbb{R}^3$. The generators of this action pushed down to the projective plane are

$$
\begin{aligned}
&\partial_p, \quad \partial_q, \quad p\partial_q, \quad q\partial_p, \quad -p^2\partial_p - pq\partial_q, \quad -pq\partial_p - q^2\partial_q, \\
&p\partial_p - q\partial_q, \quad p\partial_p + 2q\partial_q.
\end{aligned}
$$

More precisely, a vector field corresponding to an element

$$
M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & -a_{11} - a_{33} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathbf{sl}(3, \mathbb{R})
$$

is

$$
\begin{aligned}
X_M &= [a_{13} + (a_{11} - a_{33})p + a_{12}q - a_{31}p^2 - a_{32}pq]\partial_p \\
&+ [a_{23} + a_{21}p - (a_{11} + 2a_{33})q - a_{31}pq - a_{32}q^2]\partial_q.
\end{aligned} \tag{11}
$$

4 Curved metrics from the Tzitzéica equation

Consider the reduced ASDYM Lax pair (9)

$$L_0 = \partial_x - A_x - \lambda A_v, \qquad L_1 = -A_u - \lambda(\partial_y - A_y),$$

such that $[L_0, L_1] = 0$ yields (1) and use (11) to replace the matrices (10) by vector fields. Now compare the resulting Lax pair with (4), and read off the null tetrad for a hyper-complex metric (some care needs to be taken with signs because $[X_M, X_N] = -X_{[M,N]}$). This yields

$$\begin{aligned} X &= \partial_x + (-\omega_x p + pq)\partial_p + (\omega_x q - p + q^2)\partial_q, & U &= -e^\omega p^2\partial_p - e^\omega pq\partial_q, \\ Y &= \partial_y - e^{-2\omega}q\partial_p - e^\omega\partial_q, & V &= \partial_p. \end{aligned} \tag{12}$$

The first two equations in (6) are satisfied trivially, and the third one yields

$$[X, Y] - [U, V] = (\omega_{xy} + e^{-2\omega} - e^\omega)(p\partial_p - q\partial_q)$$

which is 0 if $\omega(x, y)$ satisfies equation (1). Let $p = \exp(P), q = \exp(Q)$. The frame of dual one forms is

$$\begin{aligned} e_X &= \mathrm{d}x, & e_U &= (\omega_x e^{-\omega-P} + e^{-\omega-P+Q} - e^{-\omega-Q})\mathrm{d}x - e^{-P-Q}\mathrm{d}y - e^{-\omega-P}\mathrm{d}Q, \\ e_Y &= \mathrm{d}y, & e_V &= (2\omega_x e^P - e^{2P-Q})\mathrm{d}x + (e^{Q-2\omega} - e^{\omega+P-Q})\mathrm{d}y - e^P\mathrm{d}Q + e^P\mathrm{d}P. \end{aligned} \tag{13}$$

Finally the metric is given by

$$g = 2(e_X e_Y - e_U e_V). \tag{14}$$

It is instructive to verify our calculation by considering the dual formulation of Boyer. Using the identification between the two-forms, and endomorphisms given by g define a basis $(\Omega_I, \Omega_S, \Omega_T)$ of Λ^2_+ by

$$\Omega_I(X_1, X_2) = -g(IX_1, X_2), \qquad \Omega_S(X_1, X_2) = -g(SX_1, X_2), \qquad \Omega_T(X_1, X_2) = -g(TX_1, X_2),$$

so that

$$\Omega_S = e_X \wedge e_U - e_Y \wedge e_V, \qquad \Omega_T = e_X \wedge e_Y - e_U \wedge e_V, \qquad \Omega_I = e_X \wedge e_U + e_Y \wedge e_V.$$

The Lee form $\mathcal{A}$ can be found, such that equations (8) reduce down to (1). Indeed, taking

$$\mathcal{A} = (3e^{P-Q} - 4\omega_x)\mathrm{d}x + (3e^{\omega-Q} - \omega_y)\mathrm{d}y - \mathrm{d}P + 2\mathrm{d}Q$$

yields

$$\begin{aligned} &\mathrm{d}\Omega_I - \mathcal{A} \wedge \Omega_I = 0, \\ &\mathrm{d}\Omega_S - \mathcal{A} \wedge \Omega_S = 0, \\ &\mathrm{d}\Omega_T - \mathcal{A} \wedge \Omega_T = e^\omega[\omega_{xy} + e^{-2\omega} - e^\omega]\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}(P + Q) = 0. \end{aligned}$$

The metric (14) is therefore never conformal to pseudo-hyper-Kähler because $\mathrm{d}\mathcal{A} \neq 0$. Even the simplest solution $\omega = 0$ yields a non-trivial hyper-complex structure[3]

$$\begin{aligned} g &= (e^P - e^{2P-2Q})\mathrm{d}x^2 + (3 - 2e^{P-2Q} - e^{2Q-P})\mathrm{d}x\mathrm{d}y + (e^{-P} - e^{2Q})\mathrm{d}y^2 - 2\mathrm{d}Q^2 + 2\mathrm{d}Q\mathrm{d}P \\ &+ (e^{P-Q} - e^{Q)}\mathrm{d}x\mathrm{d}P + e^{-Q}\mathrm{d}y\mathrm{d}P + (e^Q - 2e^{P-Q})\mathrm{d}x\mathrm{d}Q + (e^{Q-P} - 2e^Q)\mathrm{d}y\mathrm{d}Q. \end{aligned}$$

The Backlund transformations for the Tzitzéica equation [12, 2, 3] may now be used to generate more complicated metrics.

[3] It is worth remarking that a Tzitzéica surface corresponding to $\omega = 0$ (so called Jonas Hexenhut) is also non-trivial.

5 The twistor correspondence

From the point of view of the Yang-Mills equations, the solutions (14) that we have obtained are metrics on the total space of $\mathcal{E}$, the $\mathbb{RP}^2$-bundle associated to the Yang-Mills bundle. In this section we explain how our construction ties in with the twistor correspondences.

Consider the manifold $\mathcal{Z} = \mathbb{R}^{2,2} \times \mathbb{CP}^1$ ($\mathbb{R}^{2,2}$ denotes $\mathbb{R}^4$ with a flat metric of signature $(2,2)$). It decomposes into two open sets

$$\begin{aligned}\mathcal{Z}_+ &= \{(x^a,\lambda) \in \mathcal{Z};\ \mathrm{Im}(\lambda) > 0\} = \mathbb{R}^{2,2} \times D_+,\\ \mathcal{Z}_- &= \{(x^a,\lambda) \in \mathcal{Z};\ \mathrm{Im}(\lambda) < 0\} = \mathbb{R}^{2,2} \times D_-,\end{aligned}$$

where $D_\pm$ are two copies of a Poincare disc. These sub-manifolds are separated by

$$\mathcal{F}_0 = \{(x^a,\lambda) \in \mathcal{Z};\ \mathrm{Im}(\lambda) = 0\} = \mathbb{R}^{2,2} \times \mathbb{RP}^1.$$

The complex structures on $\mathcal{Z}_\pm$ are specified by a distribution $\mathcal{D}$ of anti-holomorphic vector fields

$$\mathcal{D} = \{\partial_x - \lambda\partial_v,\ \partial_u - \lambda\partial_y,\ \partial_{\overline{\lambda}}\}.$$

The above distribution with $\lambda \in \mathbb{RP}^1$ defines a foliation of $\mathcal{F}_0$ with a quotient $\mathcal{Z}_0$ which leads to a double fibration:

$$\mathcal{M} \xleftarrow{r} \mathcal{F}_0 \xrightarrow{s} \mathcal{Z}_0. \tag{15}$$

The *twistor space* $\mathcal{Z}$ is a three complex dimensional union of two open subsets $\mathcal{Z}_\pm$ separated by a three-dimensional real boundary (*real twistor space*) $\mathcal{Z}_0 := s(\mathcal{F}_0)$.

Each point $\mathbf{x} \in \mathbb{R}^{2,2}$ determines a holomorphic curve $L_\mathbf{x}$ made up of two sheets $D_\pm$ of complex structures (3) compactified by adding S^1:

$$\mathbf{x} = (x,y,u,v) \longrightarrow L_\mathbf{x} = \{(\omega^0,\omega^1,\lambda) : \omega^0(\lambda) = v + \lambda x,\ \omega^1(\lambda) = u + \lambda y,\ \lambda \in \mathbb{CP}^1\}$$

The normal bundle $N = T\mathcal{Z}|_{L_\mathbf{x}}/TL_\mathbf{x}$ of $L_\mathbf{x}$ in $\mathcal{Z}$ is a direct sum of two line bundles with a Chern class equal to one $\mathcal{O}(1) \oplus \mathcal{O}(1)$. If $\mathbf{x}$ and $\mathbf{x}'$ both lie on a self-dual null plane in $\mathbb{R}^{2,2}$ then $L_\mathbf{x}$ and $L_{\mathbf{x}'}$ intersect in $\mathcal{Z}$ at one point for which $\lambda \in \mathbb{RP}^1$.

Now we turn to the $SL(3,\mathbb{R})$ ASDYM equations on $\mathbb{R}^{2,2}$ with two commuting symmetries X_1, X_2. Let $\mathcal{E} = \mathbb{R}^{2,2} \times \mathbb{RP}^2$ be the bundle associated to the Yang-Mills bundle by the representation of $SL(3,\mathbb{R})$ as projective transformations of $\mathbb{RP}^2$. The $SL(3,\mathbb{R})$ ASDYM connection defines, by a $(++--)$ version of a Ward construction [13], two holomorphic vector bundles $E_{W\pm} \to \mathcal{Z}_\pm$. The following construction describes also the general case of $G =$Diff($\mathbb{RP}^2$). For this it is convenient to use the bundles $\mathcal{E}_{W\pm}$ associated to $E_{W\pm}$ by the G action on $\mathbb{RP}^2$ (the Ward bundles have infinite-dimensional fibres).

On the other hand, any pseudo-hyper-complex four-metric corresponds to a deformed twistor space $\mathcal{Z}_\mathcal{M}$, [1, 5].

Proposition 2 *Let $\mathcal{Z}_\mathcal{M}$ be a three-dimensional complex manifold with*

- *a four parameter family of rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$,*
- *a holomorphic projection $\mu : \mathcal{Z}_\mathcal{M} \longrightarrow \mathbb{CP}^1$,*
- *an anti-holomorphic involution $\rho : \mathcal{Z}_\mathcal{M} \to \mathcal{Z}_\mathcal{M}$ fixing a real equator of each rational curve.*

Then the real moduli space $\mathcal{M}$ of the ρ-invariant curves is equipped with conformal class $[g]$ of pseudo-hyper-Hermitian metrics. Conversely, given a real analytic pseudo-hyper-Hermitian metrics there exists a corresponding twistor space with the above structures.

The existence of the holomorphic projection μ reflects the fact that the Lax pair (4) for the pseudo-hyper-complex equations doesn't contain vector fields ∂_λ.

In this paper we have explained how the quotient q of $\mathcal{E}$ by lifts of X_1, X_2 is, by Proposition 4, equipped with a pseudo-hyper-complex metric . To give a more complete picture we can construct the deformed twistor space directly from $\mathcal{E}_{W\pm}$ and show that this is the twistor space of $\mathcal{M}$.

Given an analytic solution to (1) one can obtain the corresponding twistor space by equipping $\mathcal{M} \times \mathbb{CP}^1$ with a structure of a complex manifold $\mathcal{Z}$: The basis of $[0, 1]$ vectors is given by the distribution $\mathcal{D}_{\mathcal{M}}$ consisting of the Lax pair for the Tzitzéica equation together with the standard complex structure on the $\mathbb{CP}^1$. The point is that this distribution can be obtained directly from $\mathcal{D}$. To see it consider the following chain of correspondences:

$$
\begin{array}{ccccc}
\mathcal{Z}_{\mathcal{M}} = \mathcal{Z}_{\mathcal{M}-} \cup \mathcal{Z}_{\mathcal{M}0} \cup \mathcal{Z}_{\mathcal{M}+} & \stackrel{\tilde{\kappa}}{\longleftarrow} & \mathcal{E}_W = \mathbb{R}^{2,2} \times \mathbb{RP}^2 \times \mathbb{CP}^1 & \stackrel{\pi}{\longrightarrow} & \{\mathcal{Z}, \mathcal{D}\} \\
\downarrow & & \uparrow & & \downarrow \\
\mathcal{M} & \stackrel{\kappa}{\longleftarrow} & \mathcal{E} = \mathbb{R}^{2,2} \times \mathbb{RP}^2 & \longrightarrow & \mathbb{R}^{2,2}.
\end{array}
$$

Here $\mathcal{Z}$ and $\mathcal{Z}_{\mathcal{M}}$ are the twistor spaces of $\mathbb{R}^{2,2}$ and $\mathcal{M}$ respectively. The twistor space $\mathcal{Z}_{\mathcal{M}}$ is defined as the quotient $\tilde{\kappa}$ of $\mathcal{E}_W$ by lifts of symmetries X_1, X_2. The complex structures on $\mathcal{Z}_{\mathcal{M}\pm}$ are given a sub-bundle

$$\mathcal{D}_M = \tilde{\kappa}(\pi^*\mathcal{D}) = \{L_0, L_1, \partial_{\overline{\lambda}}\} \subset T\mathcal{Z}_{\mathcal{M}},$$

where

$$
\begin{aligned}
L_0 &= \partial_x + (-\omega_x p + pq)\partial_p + (\omega_x q - p + q^2)\partial_q - \lambda\partial_p \\
L_1 &= -e^{\omega}p^2\partial_p - e^{\omega}pq\partial_q - \lambda(\partial_y - e^{-2\omega}q\partial_p - e^{\omega}\partial_q).
\end{aligned}
$$

Here π is a holomorphic fibration of the associated Ward bundle. The real three-dimensional surface $\mathcal{Z}_{\mathcal{M}0} \subset \mathcal{Z}_{\mathcal{M}}$ is a quotient of $\mathbb{R}^{2,2} \times \mathbb{RP}^2 \times \mathbb{RP}^1$ by the four-dimensional real distribution $\{L_0, L_1, X_1, X_2\}$. Moreover $\mathcal{Z}_{\mathcal{M}}$ is holomorphicaly fibered over $\mathbb{CP}^1$ and it has a $\mathcal{O}(1) \oplus \mathcal{O}(1)$ rational curve embedded in it. Both structures are pulled back from $\mathcal{Z}$ and projected by $\tilde{\kappa}$. The compatibility of these projections is a consequence of the commutativity of the the above diagram, which follows from the integrability the the distribution spanned by (lifts of)

$$X_1,\ X_2,\ L_0,\ L_1,\ \partial_{\overline{\lambda}}$$

and from the fact that (X_1, X_2) commute with (L_0, L_1).

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